NOTES ON KLEE'S PAPER 'POLYHEDRAL SECTIONS OF CONVEX BODIES"

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ABSTRACT

In sections 2 and 3 two methods for proving the non existence of certain universal Banach spaces, are presented. In section 4 it is proved that every infinite-dimensional conjugate Banach space has a two-dimensional subspace whose unit cell is not a polygon.

1. **Introduction.** A Banach space X is called universal for a class of Banach spaces if every member of this class is isometric to a suitable subspace of X. S. Mazur has raised the question whether for a given integer $j \geqq 2$ there is a finite-dimensional Banach space which is universal for the class of all j-dimensional spaces. This question was solved negatively for $j = 2$ (and hence for every $j > 2$) by C. Bessaga [1]. He showed that the set of all subspaces of a finite-dimensional space cannot include all two-dimensional spaces since it has a too low dimension. (Since we do not use this fact here we do not enter into the precise definition of the notions involved in it.) Klee [4] has extended considerably the argument of Bessaga and obtained for example numerical estimates for the dimension of Banach spaces universal for all j-dimensional spaces whose unit cell is a polyhedron with r vertices.

Our first aim in this paper is to present a different method for proving the non-existence of certain kinds of universal finite-dimensional Banach spaces. Our method seems to be conceptually more elementary than that of Bessaga and Klee. We exhibit for every n a set of 2^n polyhedral 3-dimensional Banach spaces such that no Banach space of dimension $\leqq cn/\log n$ has all those spaces as subspaces. Our aim is mainly to present the method and we have not tried to get sharp estimates. From the results of Bessaga and Klee it follows by a routine compactness argument that for every j and n there are $k(j,n)$ j-dimensional spaces such that no n-dimensional space is universal for them. It seems to us that from the point of view of estimates for $k(j,n)$ our method will probably give better results than those obtainable by [4]. We have not checked

Received October 7, 1966.

* The research reported in this document has been sponsored by the Air Force Office of Scientific Research under Grant AF EOAR 66–18, through the European Office of Aerospace Research (OAR) United States Air Force.

this and moreover we are sure that also the estimates of $k(j,n)$ which can be obtained by our method are rather crude.

The method used here is closely related to the one used in [7, pp. 98–99], for proving a result on the extension of operators. As far as Mazur's problem is concerned, this method has one obvious disadvantage in comparison with Bessaga's. It cannot be used in the case $j=2$, it works only for $j \geqq 3$. It is hoped that the method can be modified so as to solve some open problems concerning infinite-dimensional universal spaces (see section 5(a)).

At the end of his paper [4] Klee mentions some infinite-dimensional questions. One of the questions (originally posed by S. Mazur) is whether there exists a separable reflexive Banach space which is universal for all separable reflexive spaces. This problem was solved negatively by W. Szlenk (private communication 1965). Szlenk showed in fact that there is no Banach space X with a separable conjugate such that every separable reflexive space is isomorphic to a subspace of X. We present here an extremely simple solution of Mazur's problem. Our method is not powerful enough to derive Szlenk's result but it applies in many situations in which Szlenk's method does not apply.

Section 4 is devoted to the solution of another problem mentioned at the end of [4]. Klee called a Banach space X polyhedral if the unit cell of every finite-dimensional subspace of X is a polyhedron. He showed that c_0 is polyhedral and asked whether there is an infinite-dimensional reflexive polyhedral space. By using a method due to Klee himself [6] we prove in section 4 that every infinite-dimensional space has a two-dimensional quotient space whose unit cell is not a polygon. Hence no infinite-dimensional conjugate space is polyhedral.

We conclude the paper with a section devoted to open problems.

All Banach spaces are taken over the reals. Let X be a Banach space. We denote by $S_X(x_0, r)$ the cell $\{x; \|x-x_0\| \leqq r\}$. The unit cell $S_X(0,1)$ is denoted also by S_X.

Section 2. We bring first three simple and well known lemmas.

LEMMA 2.1. *Let X be a Banach space and let $\{S_X(x_i, r_i)\}$ be a family (finite or infinite) of mutually intersecting cells in X. Then there is a Banach space $Y \supset X$ with $\dim Y/X = 1$ and a point $y \in Y$ such that $\|y - x_i\| \leqq r_i$ for every i.*

This lemma is due to Nachbin [8]. Two simple proofs of it may be found in [7, p. 51]. The second of these proofs (due to Grünbaum) shows that if S_X has $2h$ extreme points and if there are k given cells than Y can be chosen so that S_Y has at most $2h + 2k$ extreme points.

LEMMA 2.2. *Let X be an m-dimensional Banach space ($m < \infty$). Let $\delta > 0$ and let $\{x_i\}_{i=1}^k$ be a set of points in S_X such that $\|x_i - x_j\| \geqq 2\delta$ for $i \neq j$. Then $k \leqq (1+\delta^{-1})^m$.*

Proof. The interiors of the cells $S_X(x_i, \delta)$ are mutually disjoint and all are contained in $S_X(0, 1+\delta)$. By considering the volumes of the cells the result follows.

LEMMA 2.3. *Let $B(n)$, $n > 3$, be the two-dimensional Banach space whose unit cell is the regular $2n$-gon. Then there are vectors $\{b_i\}_{i=1}^n$ in $B(n)$ such that $\| b_i \| = 2n^2/\pi^2$ for every i and $\| b_i \pm b_j \| \leqq \| b_i \| + \| b_j \| - 2$ for $i \neq j$.*

Proof. Take n consecutive sides of the boundary of the unit cell of $B(n)$. Let b_i be the vector of norm $2n^2/\pi^2$ in the direction of the middle of the ith side. Then for $i \neq j$,

$$\| b_i \pm b_j \| \leqq 4n^2\pi^{-2}\cos^2 \pi/n \leqq 4n^2\pi^{-2} - 2.$$

We are now ready to prove

THEOREM 2.1. *Let n be an integer > 3. Then there exist 2^n three-dimensional Banach spaces $\{C_\theta\}$ such that for every θ, S_{C_θ} is a polyhedron with $4n+2$ vertices and such that there exists no m-dimensional space which is universal for all the C_θ if $2^n > 3^m(2n^2+1)^{2m}$.*

Proof. Let $B = B(n)$ and $\{b_i\}_{i=1}^n$ be as in Lemma 2.3. For every choice of n signs $\theta = (\theta_1, \theta_2, \cdots, \theta_n)$, i.e. $\theta_i = \pm 1$ for every i, there is by Lemma 2.1 a Banach space $C_\theta \supset B$ and a vector $u_\theta \in C_\theta$ such that $\| u_\theta \| \leqq 1$ and $\| u_\theta - \theta_i b_i \| \leqq \| b_i \| - 1$ for every i. By the proof due to Grünbaum of Lemma 2.1 we can take as C_θ a space whose unit cell has at most $4n+2$ extreme points. We may even assume that S_{C_θ} has exactly $4n+2$ extreme points (otherwise approximate the unit cell by a polyhedron with $4n+2$ vertices and use such an approximation as a new unit cell. It will be evident that by doing this we do not effect the argument below).

Choose in B two vectors y, z with $\| y \| = \| z \| = 1$ such that $\| \lambda y + \mu z \| \leqq \max(|\lambda|, |\mu|)$ for all λ and μ. We have that $b_i = \lambda_i y + \mu_i z$ with $|\lambda_i|, |\mu_i| \leqq 2n^2/\pi^2$ for every i.

Let X be a Banach space of dimension m which is universal for all the C_θ and consider all possible isometric embeddings of B in X. By Lemma 2.2, with $\delta = 1/(2n^2)$, there exist $k < (2n^2+1)^{2m}$ isometric operators $T_j : B \to X$, $j = 1, \cdots, k$, such that for every isometry $T : B \to X$ there is a j such that

$$\| Ty - T_jy \| \leqq n^{-2}, \quad \| Tz - T_jz \| \leqq n^{-2}. \tag{2.1}$$

Take now two different n-tuples of signs θ' and θ''. Let T' and T'' be isometries from $C_{\theta'}$ and $C_{\theta''}$ respectively into X and assume that there is a common j for which (2.1) holds for the restrictions of T' and T'' to B. Since $\theta'_i \neq \theta''_i$ for some i we may assume without loss of generality that $\theta'_1 = 1$ and $\theta''_1 = -1$. Then

$$\| T'u_{\theta'} - T_j b_1 \| \leqq \| T'u_{\theta'} - T'b_1 \| + \| T'b_1 - T_j b_1 \| \leqq$$
$$\leqq \| u_{\theta'} - b_1 \| + 2 \cdot 2n^2\pi^{-2} \cdot n^{-2} \leqq \| b_1 \| - 1 + 4\pi^{-2} < \| b_1 \| - 1/2.$$

Similarly $\| T''u_\theta + T_j b_1 \| < \| b_1 \| - 1/2$. Since $\| T_j b_1 \| = \| b_1 \|$ we get that $\| T'u_{\theta'} - T''u_{\theta''} \| > 1$. Hence, by Lemma 2.2, there are at most 3^m different θ for which there is an isometry $T_\theta : C_\theta \to X$ such that the restriction of T_θ to B satisfies (2.1) for a given j (observe that $\| T_\theta u_\theta \| \leqq 1$ for every θ and use the lemma with $\delta = 1/2$). Since the number of the indices j is $k \leqq (2n^2 + 1)^{2m}$ we get that $2^n \leqq 3^m(2n^2 + 1)^{2m}$. Q.E.D.

Section 3. We present now a simple method for proving the non-existence of certain infinite-dimensional universal spaces. If X and Y are Banach spaces and $1 \leqq p \leqq \infty$ we denote by $(X \oplus Y)_p$ the space of all pairs (x, y) with $\| (x, y) \| = (\| x \|^p + \| y \|^p)^{1/p}$ if $p < \infty$ and $= \max(\| x \|, \| y \|)$ if $p = \infty$ ($x \in X$ and $y \in Y$). The one dimensional space is denoted by R.

LEMMA 3.1. *Let X be a Banach space and let $1 \leqq p \leqq \infty$. Assume that $(X \oplus R)_p$ is isometric to a subspace of X. Then X has a subspace isometric to l_p if $p < \infty$ and to c_0 if $p = \infty$.*

Proof. We consider the case $p < \infty$ only, the proof for $p = \infty$ is similar. By our assumption X contains a vector x_1 with $\| x_1 \| = 1$ and a subspace Y_1 which is isometric to X such that $\| x_1 + y \|^p = 1 + \| y \|^p$ for every $y \in Y_1$. Continuing inductively we get for every n a vector x_n in Y_{n-1} with $\| x_n \| = 1$ and a subspace Y_n of Y_{n-1} which is isometric to X and such that $\| x_n + y \|^p = 1 + \| y \|^p$ for every $y \in Y_n$. It follows easily that for every real $\{\lambda_i\}_{i=1}^n$ we have $\| \sum_{i=1}^n \lambda_i x_i \|^p = \sum_{i=1}^n |\lambda_i|^p$. Hence the subspace of X spanned by $\{x_i\}_{i=1}^\infty$ has the desired properties.

THEOREM 3.1. *The following sets of Banach spaces do not have a member which is universal for all the spaces in this set.*

(i) *All reflexive spaces of a given density character $\mathcal{M}$.*

(ii) *All spaces whose conjugate is separable.*

(iii) *All conjugate separable spaces.*

(iv) *All spaces of a given density character $\mathcal{M}$ which do not contain an infinite-dimensional reflexive subspace.*

Proof. All these facts follow easily from Lemma 3.1. (i) follows since c_0 and l_1 are not reflexive. (ii) is a consequence of the fact that l_1^* is not separable. To derive (iii) we have to use a result of Bessaga and Pełczyński [2] that no separable conjugate space contains a subspace isomorphic to c_0. Finally, (iv) follows by using the lemma for $1 < p < \infty$.

Section 4. In this section we prove that there is no conjugate infinite-dimensional polyhedral space. First some notations and conventions. Let P be a symmetric convex body in the plane R^2. The boundary of P will be denoted by ∂P and for $0 < \varepsilon < 1$ the interior of the set $(1+\varepsilon)P \sim (1-\varepsilon)P$ will be denoted by $\varepsilon - nbd\,\partial P$. We shall consider in the sequel many norms in R^2. However, for the sake of definiteness we single out one norm (an inner product norm, say) and this will be used whenever the symbol $\| \cdot \|$ is used for a vector in R^2 or an operator into R^2. All other norms in the plane will appear only implicitly through their unit cell and the symbol $\| \cdot \|$ will not be used for them. If $Y \subset X$ and T is an operator on X its restriction to Y will be denoted by $T|_Y$.

LEMMA 4.1. *Let X be a finite-dimensional Banach space whose unit cell is a polyhedron. Let $\varepsilon > 0$ and let T be a linear operator from X onto R^2. Assume that there is a subspace Y of X of co-dimension 1 such that $TS_X = TS_Y$. Then there is an operator $\tilde{T}: X \to R^2$ such that $\| T|_Y - \tilde{T}|_Y \| \leqq \varepsilon$, $\partial \tilde{T} S_X \subset \varepsilon - nbd\partial TS_X$ and such that $\tilde{T}S_X$ has more extreme points than TS_X.*

Proof. Let $\{\pm c_i\}_{i=1}^n$ be the extreme points of TS_X and put $A_i = T^{-1}c_i \cap S_X$, $i = 1,2,\cdots,n$. Assume first that there is an index $i = i_0$ such that A_{i_0} consists of more than one point. Let $f \in X^*$ be such that f takes on A_{i_0} positive and negative values. Let $c \in R^2$ be such that $\{c_{i_0} + \lambda c; \lambda \in R\} \cap TS_X = c_{i_0}$ and put $T_\theta x = Tx + \theta f(x)c$. It is easily verified that for small enough θ the operator T_θ has all the properties required from $\tilde{T}$.

Assume now that A_i is a single point for every i. By our assumptions $A_i \in Y$ for every i. For $\delta > 0$, let T_δ be an operator from X into R^2 such that T_δ is one to one on $\operatorname{ext} S_X$ and $\| T_\delta - T \| < \delta$. It is easily checked that for small enough δ the points $\{\pm T_\delta A_i\}_{i=1}^n$ are extreme points of $T_\delta S_X$. Take one such $\delta = \delta_0 < \varepsilon/2$. If $T_{\delta_0}S_X$ has more than $2n$ extreme points we can take $\tilde{T} = T_{\delta_0}$. If $\{\pm T_{\delta_0}A_i\}_{i=1}^n$ are all the extreme points of $T_{\delta_0}S_X$ then by the argument of Klee in [6] there is a $\tilde{T}: X \to R^2$ such that $\tilde{T}|_Y = T_{\delta_0}|_Y$, $\tilde{T}S_X$ has more than $2n$ extreme points and $\partial \tilde{T} S_X \subset \varepsilon/2 - nbd\partial T_{\delta_0}S_X$. This operator $\tilde{T}$ has all the desired properties.

LEMMA 4.2. *Let P be a symmetric polygon in the plane. Then there is $\varepsilon > 0$ such that every symmetric convex body C in the plane for which $\partial C \subset \varepsilon - nbd\partial P$ has at least as many extreme points as P.*

Proof. Obvious.

THEOREM 4.1. *Let X be an infinite-dimensional Banach space. Than X has a two-dimensional quotient space whose unit cell is not a polygon.*

Proof. We assume that X is an infinite-dimensional Banach space all whose two-dimensional quotient spaces are polygonal and argue to a contradiction. By the results of [5] every finite-dimensional quotient space of X is polyhedral.

Let T_1 be a bounded linear operator from X onto R^2 and let $\overline{T_1 S_X}$ have $2n_1$ extreme points. Let $T_1': X \to R^{n_1+1}$ and $U_1: R^{n_1+1} \to R^2$ be linear operators such that T_1' is bounded and onto and such that $T_1 = U_1 T_1'$. Let Z_1 be the $(n_1 + 1)$-dimensional Banach space whose unit cell is $\overline{T_1' S_X}$. The operator U_1 maps S_{z_1} onto $\overline{T_1 S_X}$ and by the choice of n_1 there is subspace Y_1 of Z_1 of codimension 1 such that $U_1 S_{Y_1}$, contains all the extreme points of $\overline{T_1 S_X}$ and hence all of $\overline{T_1 S_X}$. Let now $\varepsilon_1 > 0$ be such that Lemma 4.2 holds with $2\varepsilon_1$ if $\overline{T_1 S_X}$ is taken as P. Choose next points $\{x_i^1\}_{i=1}^{n_1}$ in S_X and a $\delta_1 > 0$ such that $T_1' x_i^1 \in Y_1$ for every i and such that whenever $\|c_i - T_1 x_i^1\| \leqq \delta_1$ for every i the convex hull C of $\{\pm c_i\}_{i=1}^{n_1}$ satisfies $\partial C \subset \varepsilon_1 - nbd\partial \overline{T_1 S_X}$. (Remark: unless X is reflexive we do not have in general that $T_1' S_X = \overline{T_1' S_X}$ and hence we cannot insure the existence of points $x_i^1 \in S_X$ such that $T_1 x_i^1 \in \text{ext}\ \overline{T_1 S_X}$. But it suffices for our purpose to choose the x_i^1 so that the $T_1 x_i^1$ are near the extreme points of $\overline{T_1 S_X}$).

By Lemma 4.1 there is an operator $\tilde{U}_1 : R^{n_1+1} \to R^2$ such that $\tilde{U}_1 \overline{T_1' S_X}$ has $2n_2 > 2n_1$ extreme points, $\|\tilde{U}_1 T_1' x_i^1 - T_1 x_i^1\| < \delta_1/2$ for every i and $\partial \tilde{U}_1 \overline{T_1' S_X} \subset \varepsilon_1 - nbd\partial \overline{T_1 S_X}$. Put $T_2 = \tilde{U}_1 T_1'$.

Next we choose $T_2' : X \to R^{n_2+n_1+1}$ and $U_2 : R^{n_2+n_1+1} \to R^2$ such that $T_2 = U_2 T_2'$. We take Y_2, a subspace of deficiency 1 in Z_2 ($= R^{n_2+n_1+1}$ with unit cell $\overline{T_2' S_X}$) such that $T_2' x_i^1 \in Y_2$ for every i and $U_2 S_{Y_2} = U_2 S_{Z_2}$. Let $\varepsilon_2 > 0$ be such that Lemma 4.2 holds for $2\varepsilon_2$, with $P = \overline{T_2 S_X}$, and such that $\varepsilon_2 - nbd\overline{T_2 S_X} < \varepsilon_1 - nbd\partial \overline{T_1 S_X}$. We choose $\delta_2 > 0$ and $\{x_i^2\}_{i=1}^{n_2} \subset S_X$ as in the first step and then (by Lemma 4.1) we choose a $\tilde{U}_2$ such that $\|\tilde{U}_2 T_2 x_i^k - T_2 x_i^k\| < \delta_i/2^2$ for $k = 1,2$, and $i = 1, \cdots, n_k$, $\partial \tilde{U}_2 \overline{T_2 S_X} \subset \varepsilon_2 - nbd\partial \overline{T_2 S_X}$ and such that the number $2n_3$ of extreme points of $\tilde{U}_2 \overline{T_2' S_X}$ is greater than $2n_2$.

Continuing we get sequences $\{n_k\}$, $\{\delta_k\}$, $\{\varepsilon_k\}$, $\{T_k\}$ and $(x_i^k\}$ such that

(4.1) $$n_1 < n_2 < \cdots < n_k < \cdots.$$

(4.2) $2n_k$ is the number of extreme points of $\overline{T_k S_X}$.

(4.3) $$\varepsilon_k\text{-}nbd\partial\overline{T_k S_k} \subset \varepsilon_{k-1}\text{-}nbd\partial\overline{T_{k-1} S_{k-1}},\ \varepsilon_k \downarrow 0.$$

(4.4) For every symmetric convex body C in R^2 such that $\partial C \subset 2\varepsilon_k\text{-}nbd\partial\overline{T_k S_k}$ the number of extreme points of C is $\geqq 2n_k$.

(4.5) $\{x_i^k\}_{i=1}^{n_k} \subset S_X$ and whenever $\|c_i - T_k x_i^k\| < \delta_k$ for $i = 1, \cdots, n_k$ we have ∂ (convex hull of $\{\pm c_i\}_{i=1}^{nk}) \subset \varepsilon_k\text{-}nbd\,\partial\overline{T_k S_k}$.

(4.6) $$\|T_k x_i^j - T_{k-1} x_i^j\| < \delta_j/2^k,\ i = 1, \cdots, n_j,\ j = 1, 2, \cdots, k-1.$$

By (4.3) $\|T_k\| \leqq (1 + \varepsilon_1)\|T_1\|$ for every k and hence by the w^* compactness of the unit cell of X^* the sequence $\{T_k\}$ has a limit point T in the strong ($=$ weak in our case) operator topology. By using (4.3) again we get that $TS_X \subset (1 + \varepsilon_k) T_k S_X$ for every k. Also, by (4.6), $\|Tx_i^k - T_k x_i^k\| < \delta_k$ for every i and k and hence by (4.5) $\overline{TS_X} \supset$ convex hull of $\{\pm Tx_i^k\}_{i=1}^{nk} \supset (1 - \varepsilon_k)\overline{T_k S_X}$. We have thus that $\partial\overline{TS_X} < 2\varepsilon_k\text{-}nbd\partial\overline{T_k S_X}$ for every k and therefore, by (4.1) and (4.4), $\overline{TS_X}$ is not a polygon. Q.E.D.

COROLLARY. *No infinite-dimensional conjugate (and in particular reflexive) Banach space is polyhedral.*

Section 5. Open problems and remarks. (a). The first problem we mention is one which was raised already in [4]. Does there exist a separable reflexive space which is universal for all finite-dimensional Banach spaces? Klee observed in [4] that the separable reflexive space $(\sum_{n=1}^{\infty} \oplus l_n^{\infty})_2$ is universal for all polyhedral finite-dimensional spaces and hence, in an obvious sense, "almost" universal for all finite-dimensional spaces. It follows that in order to establish the nonexistence (or, of course, the existence) of such a universal space, non polyhedral finite-dimensional spaces must be considered.

The method of Section 2 may be helpful in this direction. In fact, given any two-dimensional non-polyhedral Banach space B then, as is easily seen, there is an unbounded sequence $\{b_i\}_{i=1}^{\infty}$ in B such that $\| b_i \pm b_j \| \leqq \| b_i \| + \| b_j \| - 2$ for every $i \neq j$. By Lemma 2.1 there is for every sequence of signs $\theta = (\theta_1, \theta_2, \theta_3, \cdots)$ a three-dimensional space C_θ containing B and a vector $u_\theta \in C_\theta$ such that $\| u_\theta \| = 1$ and $\| u_\theta - \theta_i b_i \| \leqq \| b_i \| - 1$ for every i. It is easily verified that if for $\theta' \neq \theta''$ there are isometries T' and T'' from $C_{\theta'}$ and $C_{\theta''}$ into a Banach space X such that $T'|_B = T''|_B$ then $\| T'u_{\theta'} - T''u_{\theta''} \| = 2$. Thus if X is separable there are for every isometry $T: B \to X$ at most a countable number of sequences θ for which there is an isometry $T_\theta: C_\theta \to X$ with $T_\theta|_B = T$. It is hoped that an argument of this type suitably combined with a compactness argument will prove that there is no separable reflexive (or even conjugate) space which is universal for all three-dimensional spaces. It is conceivable that here there is a real difference between the two and three dimensional cases.

(b) Can the method of Section 3 be modified so as to establish the nonexistence of universal spaces with respect to isomorphism and not only isometry? More specifically: let X be a Banach space, let $1 \leqq p \leqq \infty$ and let $M < \infty$. Assume that for every k there is an operator

$$T_k: \overbrace{(X \oplus \cdots \oplus X)}^{k}_p \to X$$

with $\| y \| \leqq \| T_k y \| \leqq M \| y \|$ for every $y \in (X \oplus \cdots \oplus X)_p$. Must X have a subspace isomorphic to l_p if $p < \infty$ or to c_0 if $p = \infty$? Can one prove at least that X is not reflexive if $p = 1$ or ∞ ?

(c) Here are some problems concerning the existence of universal spaces in certain classes of Banach spaces.

(i) Does there exist a separable strictly convex space which is universal for all strictly convex separable spaces?

(ii) Does there exist a separable space with an unconditional basis which is universal (in the sense of isometry or isomorphism) for all separable Banach spaces with an unconditional basis?

These questions are of the type appearing in Theorem 3.1, yet our method does not seem to provide any information concerning them.

(d) Another kind of problem concerning infinite-dimensional universal spaces is exemplified by the following questions.

(i) Does there exist a separable reflexive space which is universal for all l_p, $1 < p < \infty$?

(ii) Does there exist a uniformly convex separable space which is universal for all l_p, $p_0 < p < p_1$ (with $1 < p_0 < p_1 < \infty$)?

Here it is important to use the term universal as in the introduction (i.e. in the sense of isometry). The corresponding problems for isomorphism have a positive answer (see [3] and [9]).

(e) The following question arises naturally in view of the Corollary to Theorem 4.1. Does there exist a polyhedral infinite-dimensional Banach space whose unit cell is the closed convex hull of its extreme points?

There exist polyhedral Banach spaces whose unit cell has an infinite number of extreme points. Let X be the space c_0 of the sequences $x = (x_1, x_2, \cdots)$ converging to 0 and with norm $|||\cdot|||$ whose unit cell is the convex hull of $\{x; \max_{1 \leqq i \leqq \infty} |x_i| \leqq 1\} \cup \{x; \sum_{i=1}^{\infty} |x_i| \leqq 2\}$. Then X^* is isometric to l_1 with norm

$$|||(y_1, y_2, \cdots, y_n, \cdots)||| = \max \left(\sum_{i=1}^{\infty} |y_1|, \max_{1 \leq i \leq \infty} 2|y_i| \right).$$

It is easily verified that $\operatorname{ext} S_{X^*}$ consists of those $y = (y_1, y_2, \cdots)$ for which $|y_i| = \frac{1}{2}$ for two indices i and $= 0$ for all other indices. By using the simple criterion given in [7, p. 102] it follows now easily that X is polyhedral. Obviously S_X has an infinite number of extreme points.

Let us also mention the following fact. If X and Y are polyhedral spaces then $(X \oplus Y)_1$ is also polyhedral. Since l_1 is not polyhedral it follows by Lemma 3.1 that for every cardinal number $\mathscr{M}$ there is no universal polyhedral space of density character $\mathscr{M}$.

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